CALCULUS

DS1103

CHAPTER - 01

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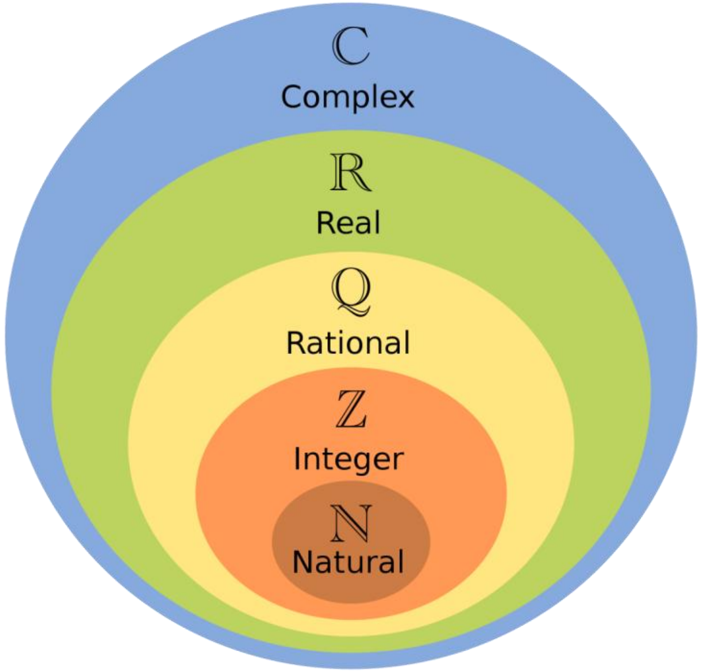
The real number system

Numbers that can be represented on a number line are called real numbers. Set of real numbers is denoted by $\mathbb{R}$.



$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$$

Axioms of the Set $\mathbb{R}$

Field Axioms:

A set $\mathbb{R}$ that has more than one element is said to be a field under two compositions of Addition and Multiplication defined in it if the following properties are satisfied for all $a, b, c \in \mathbb{R}$.

Name	Addition	Multiplication
Closure	$a, b \in \mathbb{R} \Rightarrow a + b \in \mathbb{R}$	$a, b \in \mathbb{R} \Rightarrow ab \in \mathbb{R}$
Associativity	$(a + b) + c = a + (b + c)$	$(ab)c = a(bc)$
Identity	$a + 0 = 0 + a = a$	$a1 = 1a = a$
Inverse	$a + (-a) = (-a) + a = 0$	$aa^{-1} = a^{-1}a = 1$, if $a \neq 0$
Commutativity	$a + b = b + a$	$ab = ba$
Distributivity	$a(b + c) = ab + ac$	

Order Axioms:

Generally, the order relation does not exists between the members of a set or filed. That means we cannot speak of one member being greater than or less than the other. A field is said to be an ordered field if it satisfies the following properties.

Reflexivity	$a \leq a$
Antisymmetry	$a \leq b$ and $b \leq a \Rightarrow a = b$
Transitivity	$a \leq b$ and $b \leq c \Rightarrow a \leq c$
Trichotomy	Either $a < b$ or $a = b$ or $a > b$
	$a \leq b \Rightarrow a + c \leq b + c$; $\quad a \leq b$ and $c \geq 0 \Rightarrow ac \leq bc$

❖ It can be easily seen that the set $\mathbb{Q}$ and $\mathbb{R}$ are ordered fields while the set $\mathbb{N}$ and $\mathbb{Z}$ are not fields.

Definition :

- ❖ Let S be a subset of $\mathbb{R}$. If there exists a real number m such that $m \geq s$ for all s ∈ S, then m is called an **upper bound** for S, and we say that S is bounded above.

  If $m \leq s$ for all s ∈ S, then m is a **lower bound** for S and S is bounded below.

  The set S is said to be **bounded** if it is bounded above and bounded below.

- ❖ If an upper bound m for S is a member of S, then m is called the **maximum** (or largest element) of S, and we write

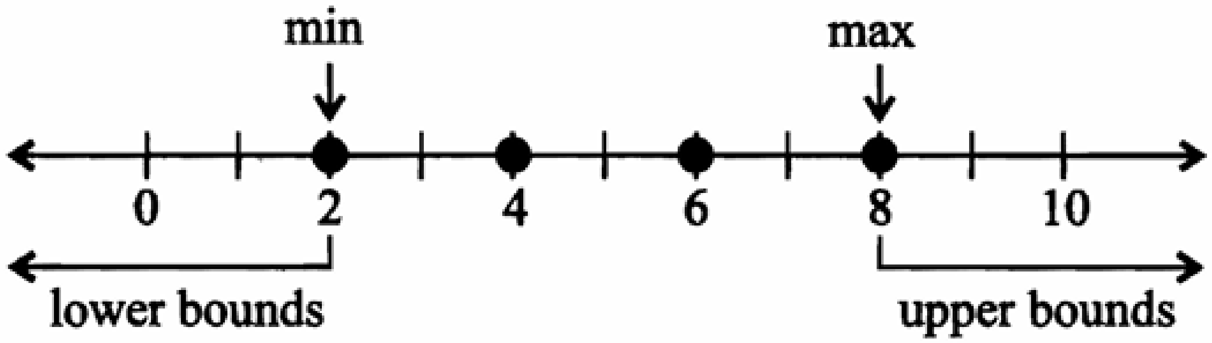
$$\boldsymbol{m = \max S}$$

  Similarly, if a lower bound of S is a member of S, then it is called the **minimum** (or least element) of S, denoted by **min S**.

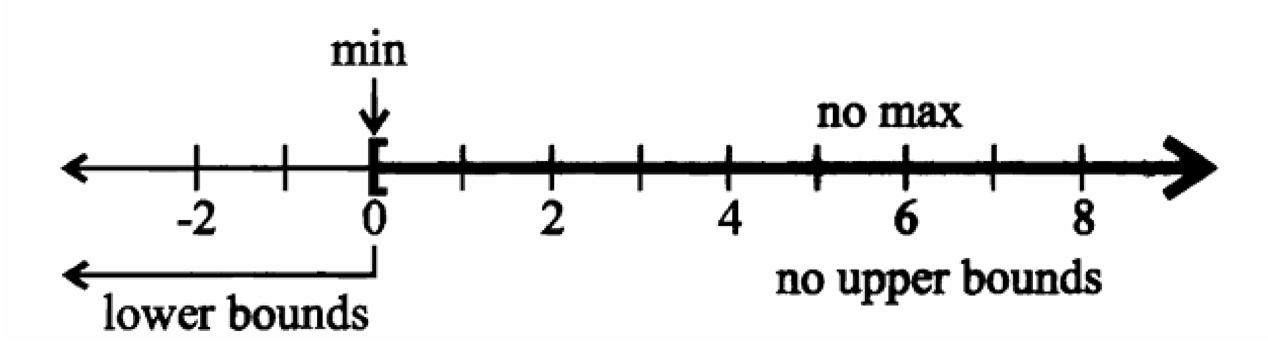
- A set may have upper or lower bounds, or it may have neither. If m is an upper bound for S, then any number grater than m is also an upper bound. While a set may have many upper and lower bounds, if it has a maximum or a minimum, then those values are unique.

Example: Let $A = (-\infty, 3)$. The set A is bounded above by 5, and it is not bounded below.

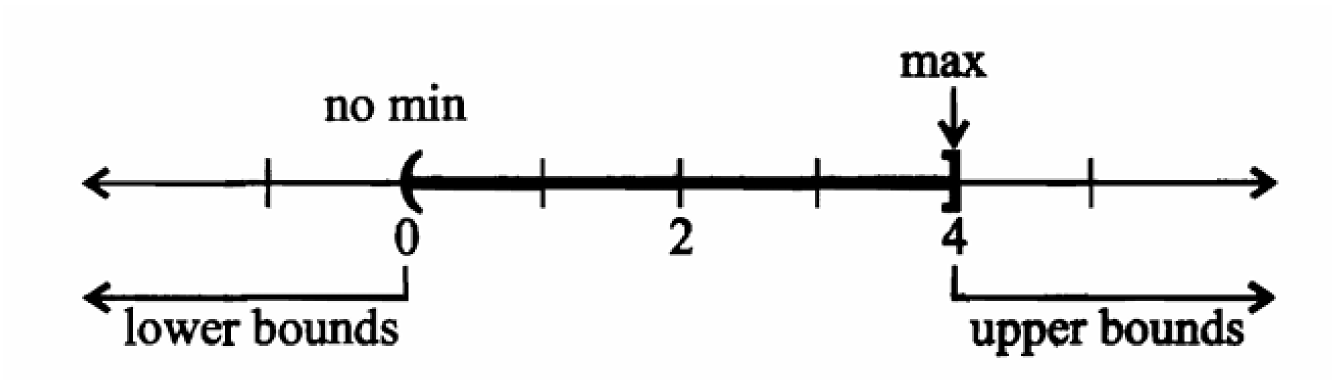
Example: The set S = {2,4,6,8} is bounded above by 8 and any other real number grater than or equal to 8.



Example: The interval $[0, \infty)$ is not bonded above.



Example: The interval $(0,4]$ has a maximum of 4, and this is the smallest of the upper bounds.



Example: Find upper and lower bounds, the maximum, and the minimum of the set $T = \{ q \in \mathbb{Q} : 0 \leq q \leq \sqrt{2} \}$, if they exist.

Definition: Let S be a nonempty subset of $\mathbb{R}$.

(a) If S is bounded above and the set of all upper bounds of S has minimum, then we say that S has a least upper bound (l.u.b.) and the smallest upper bound is called the **supremum** of S and denote it by ***sup S***.

(b) If S is bounded below and the set of all lower bounds S maximum, then we say that S has a greatest lower bound (g.l.b) and the smallest lower bound is called the **infimum** of S and denote it by $\boldsymbol{inf S}$.

Clearly, $sup\ S$ and $inf\ S$ may or may not exist and in case exist, it may or may not belong to S.

Example: Find maximum and minimum, the supremum, and the infimum of the set $A = \{\frac{1}{n^2} : n \in \mathbb{N} \ and \ n \geq 3\}$, if they exist.

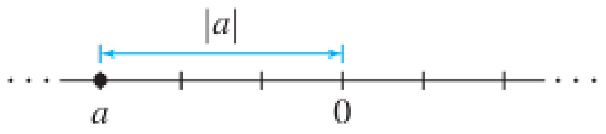
Example: Find maximum and minimum, the supremum, and the infimum of the set $D = \{x \in \mathbb{R} \ and \ x^2 < 10\}$, if they exist.

Example: Find maximum and minimum of the sets $\mathbb{N}$, $\mathbb{Q}$ and $\mathbb{Z}$, if they exist.

Definition : If $x \in \mathbb{R}$, then the **absolute value** of x, denoted by $|x|$, is defined by

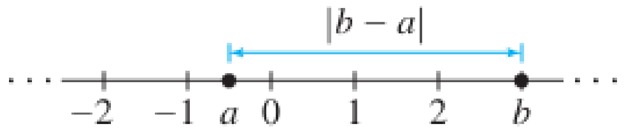
$$|x| = \begin{cases} x, & \text{if } x \geq 0, \\ -x, & \text{if } x < 0. \end{cases}$$

The basic properties of absolute value are summarized in the following theorem.



Theorem : Let $x, y \in \mathbb{R}$ and let $a \geq 0$. Then

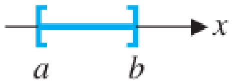
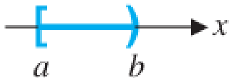
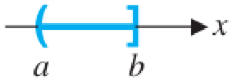
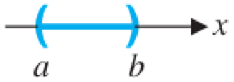
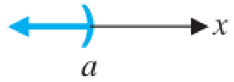
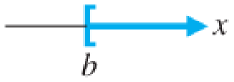
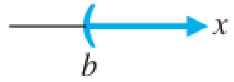
(a) $|x| \geq 0$,
(b) $|x| \leq a$ iff $-a \leq x \leq a$,
(c) $|xy| = |x| \cdot |y|$,
(d) $|x+y| \leq |x| + |y|$.



- We use standard notation for intervals. Given real numbers a > b, there are four intervals with endpoints a and b.

Closed interval $[a, b]$ (endpoints included) | Open interval (a, b) (endpoints excluded) | Half-open interval $[a, b)$ | Half-open interval $(a, b]$

$[a, b] = \{x \in \mathbf{R} : a \le x \le b\}$ $(a, b) = \{x : a < x < b\}$ $[a, b) = \{x : a \le x < b\}$ $(a, b] = \{x : a < x \le b\}$

Interval Notation	Inequality Notation	Line Graph
$[a, b]$	$a \le x \le b$	
$[a, b)$	$a \le x < b$	
$(a, b]$	$a < x \le b$	
(a, b)	$a < x < b$	
$(-\infty, a]$	$x \le a$	
$(-\infty, a)$	$x < a$	
$[b, \infty)$	$x \ge b$	
(b, ∞)	$x > b$	

❖ Open and closed intervals may be described by inequalities. For example, the interval $(-r, r)$ is described by the inequality $|x| < r$.

$$|x| < r \quad \Leftrightarrow \quad -r < x < r \quad \Leftrightarrow \quad x \in (-r, r)$$

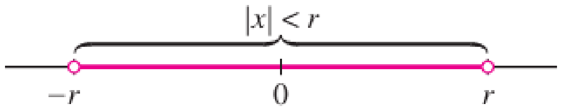


FIGURE The interval $(-r, r) = \{x : |x| < r\}$.

More generally, for an interval symmetric about the value c,

$$|x - c| < r \quad \Leftrightarrow \quad c - r < x < c + r \quad \Leftrightarrow \quad x \in (c - r, c + r)$$

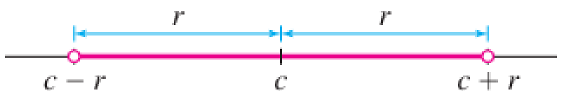


FIGURE $(a, b) = (c - r, c + r)$, where

$$c = \frac{a + b}{2}, \qquad r = \frac{b - a}{2}$$

Closed intervals are similar, with $<$ replaced by $\le$. We refer to r as the **radius** and to c as the **midpoint** or **center**. The intervals (a, b) and $[a, b]$ have midpoint $c = \frac{1}{2}(a + b)$ and radius $r = \frac{1}{2}(b - a)$

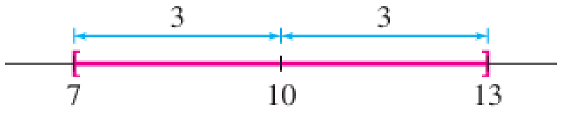


FIGURE The interval $[7, 13]$ is described by $|x - 10| \le 3$.

EXAMPLE Describe [7, 13] using inequalities.

Solution The midpoint of the interval [7, 13] is $c = \frac{1}{2}(7 + 13) = 10$ and its radius is $r = \frac{1}{2}(13 - 7) = 3$ (Figure). Therefore,

$$[7, 13] = \{x \in \mathbf{R} : |x - 10| \le 3\}$$

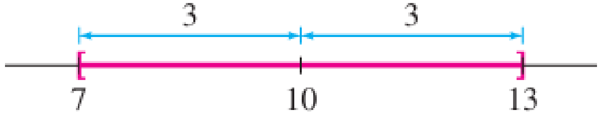


FIGURE The interval [7, 13] is described by $|x - 10| \le 3$.

EXAMPLE Describe the set $S = \{x : |\frac{1}{2}x - 3| > 4\}$ in terms of intervals.

Solution It is easier to consider the opposite inequality $|\frac{1}{2}x - 3| \le 4$ first.

$$\left|\frac{1}{2}x - 3\right| \le 4 \quad \Leftrightarrow \quad -4 \le \frac{1}{2}x - 3 \le 4$$

$$-1 \le \frac{1}{2}x \le 7 \qquad \text{(add 3)}$$

$$-2 \le x \le 14 \qquad \text{(multiply by 2)}$$

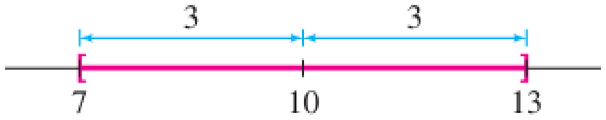


FIGURE The interval [7, 13] is described by $|x - 10| \le 3$.

Thus, $|\frac{1}{2}x - 3| \le 4$ is satisfied when x belongs to $[-2, 14]$. The set S is the *complement*, consisting of all numbers x *not in* $[-2, 14]$. We can describe S as the union of two intervals: $S = (-\infty, -2) \cup (14, \infty)$ (Figure). ■

DEFINITION A **function** f from a set D to a set Y is a rule that assigns, to each element x in D, a unique element $y = f(x)$ in Y. We write

$$f : D \to Y$$

The set D, called the **domain** of f, is the set of "allowable inputs." For $x \in D$, $f(x)$ is called the **value** of f at x (Figure). The **range** R of f is the subset of Y consisting of all values $f(x)$:

$$R = \{y \in Y : f(x) = y \text{ for some } x \in D\}$$

Informally, we think of f as a "machine" that produces an output y for every input x in the domain D (Figure).

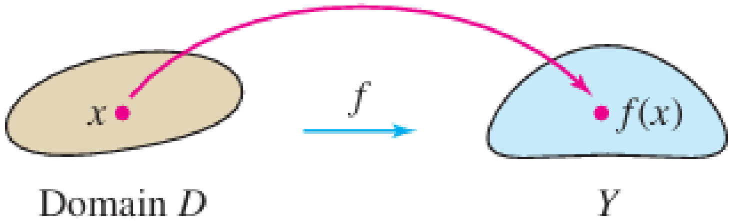


FIGURE A function assigns an element $f(x)$ in Y to each $x \in D$.

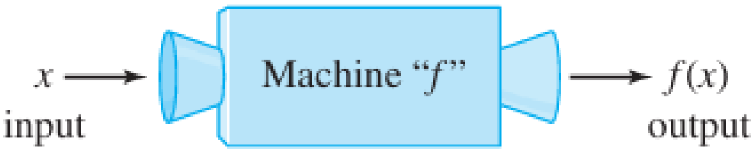


FIGURE Think of f as a "machine" that takes the input x and produces the output $f(x)$.

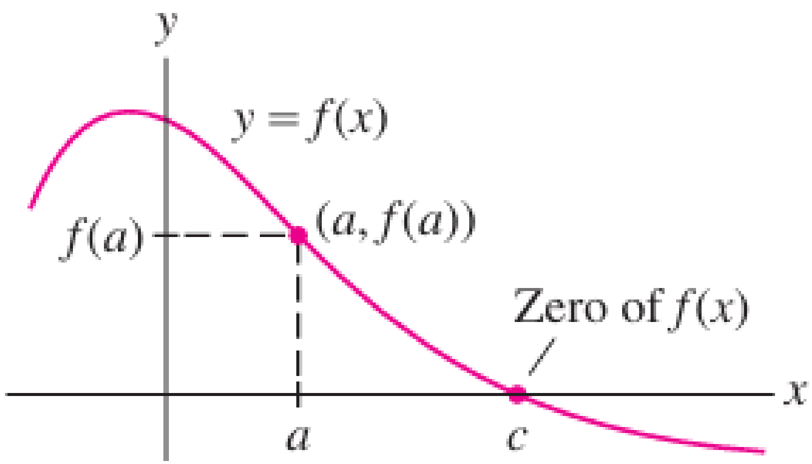
❖ The first part of this text deals with numerical functions f, where both the domain and the range are sets of real numbers. We refer to such a functions interchangeably a f or f(x).The letter x is used often to denote the independent variable that can take on any value in the domain D. We write y = f(x) and refer to y as the dependent variable.

When f is defined by a formula, its natural domain is the set of real numbers x for which the formula is meaningful. For example, the function $f(x) = \sqrt{9-x}$ has domain $D = \{x : x \leq 9\}$ because $\sqrt{9-x}$ is defined if $9 - x \geq 0$. Here are some other examples of domains and ranges:

$f(x)$	**Domain** D	**Range** R
x^2	$\mathbf{R}$	$\{y : y \geq 0\}$
$\cos x$	$\mathbf{R}$	$\{y : -1 \leq y \leq 1\}$
$\frac{1}{x+1}$	$\{x : x \neq -1\}$	$\{y : y \neq 0\}$

❖ The **graph** of a function $y = f(x)$ is obtained by plotting the points $(a, f(a))$ for a in the domain D (Figure). If you start at $x = a$ on the x-axis, move up to the graph and then over to the y-axis, you arrive at the value $f(a)$. The absolute value $|f(a)|$ is the distance from the graph to the x-axis.

A **zero** or **root** of a function $f(x)$ is a number c such that $f(c) = 0$. The zeros are the values of x where the graph intersects the x-axis.



■ **EXAMPLE** Find the roots and sketch the graph of $f(x) = x^3 - 2x$.

Solution First, we solve

$$x^3 - 2x = x(x^2 - 2) = 0.$$

The roots of $f(x)$ are $x = 0$ and $x = \pm\sqrt{2}$. To sketch the graph, we plot the roots and a few values listed in Table and join them by a curve (Figure).

TABLE 1

x	$x^3 - 2x$
−2	−4
−1	1
0	0
1	−1
2	4

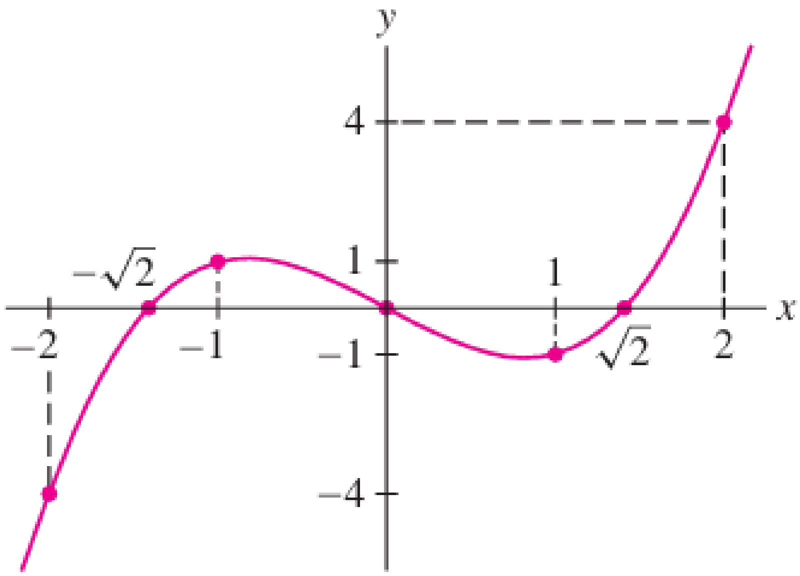


FIGURE Graph of $f(x) = x^3 - 2x$.

❖ **Vertical Line Test :** A curve in the plane is the graph of a function if and only if each vertical line x = a intersects the curve in at most one point.

We can graph not just functions but, more generally, any equation relating y and x. Figure shows the graph of the equation $4y^2 - x^3 = 3$; it consists of all pairs (x, y) satisfying the equation. This curve is not the graph of a function because some x-values are associated with two y-values. For example, $x = 1$ is associated with $y = \pm 1$. A curve is the graph of a function if and only if it passes the **Vertical Line Test**; that is, every vertical line $x = a$ intersects the curve in at most one point.

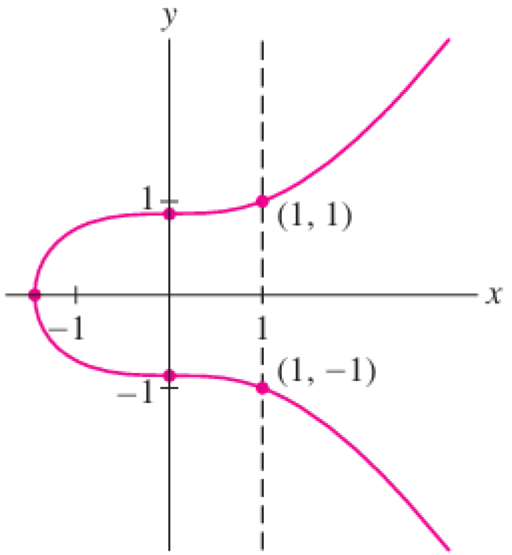
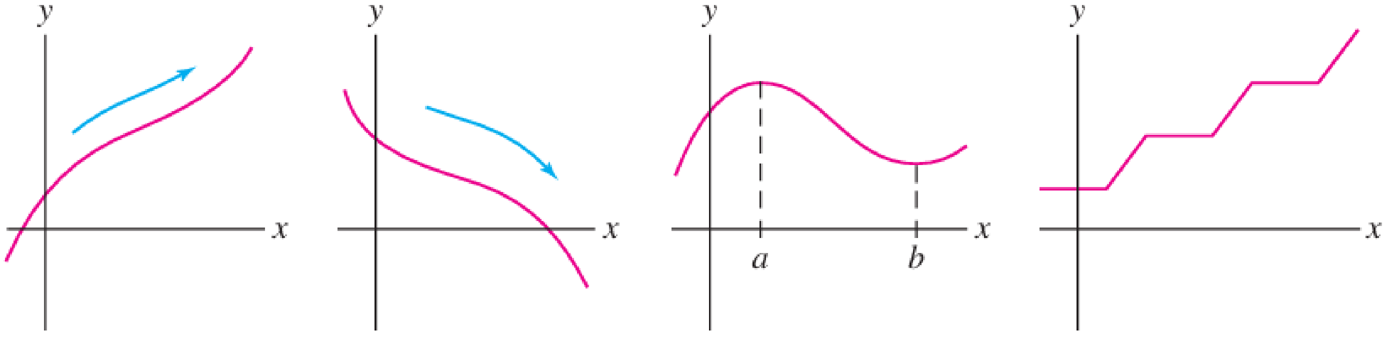


FIGURE Graph of $4y^2 - x^3 = 3$. This graph fails the Vertical Line Test, so it is not the graph of a function.

❖ Increasing and Decreasing Functions

Increasing:	$f(x_1) < f(x_2)$ if $x_1 < x_2$
Nondecreasing:	$f(x_1) \le f(x_2)$ if $x_1 < x_2$
Decreasing:	$f(x_1) > f(x_2)$ if $x_1 < x_2$
Nonincreasing:	$f(x_1) \ge f(x_2)$ if $x_1 < x_2$

We say that $f(x)$ is **monotonic** if it is either increasing or decreasing. In Figure (C), the function is not monotonic because it is neither increasing nor decreasing for all x.



(A) Increasing

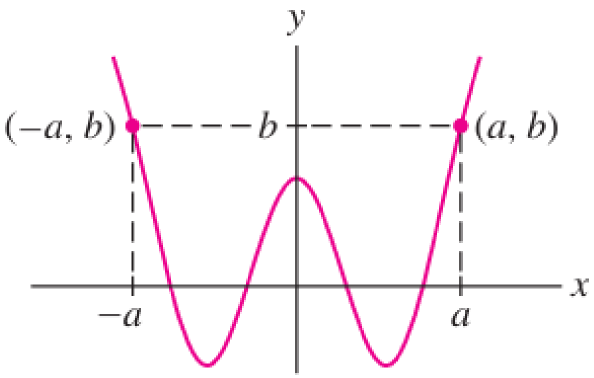
(B) Decreasing

(C) Decreasing on (a, b) but not decreasing everywhere

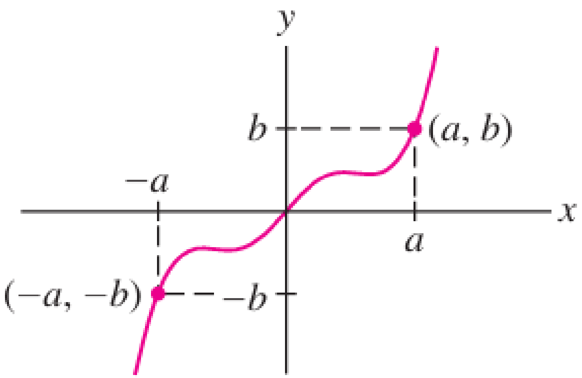
(D) Nondecreasing but not increasing

❖ Even function and Odd function

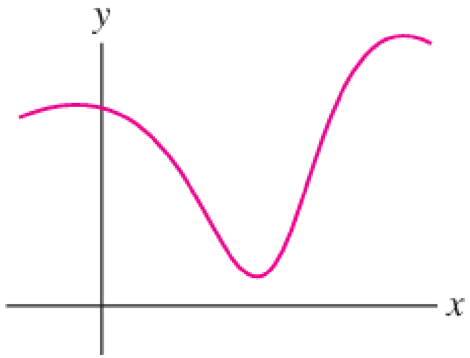
- $f(x)$ is **even** if $f(-x) = f(x)$
- $f(x)$ is **odd** if $f(-x) = -f(x)$



(A) Even function: $f(-x) = f(x)$
Graph is symmetric about the y-axis.



(B) Odd function: $f(-x) = -f(x)$
Graph is symmetric about the origin.



(C) Neither even nor odd

EXAMPLE Determine whether the function is even, odd, or neither.

(a) $f(x) = x^4$ **(b)** $g(x) = x^{-1}$ **(c)** $h(x) = x^2 + x$

Solution

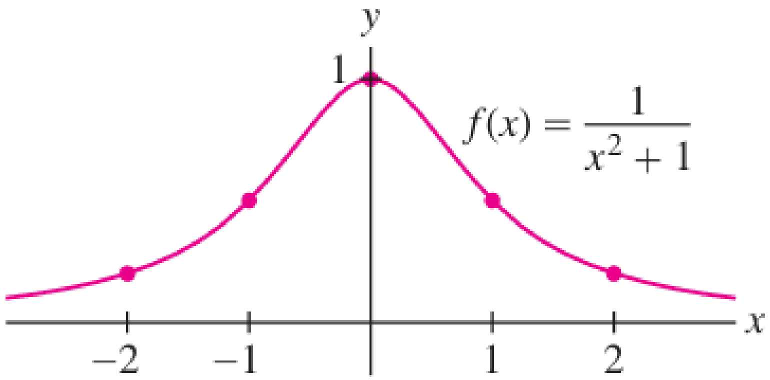
(a) $f(-x) = (-x)^4 = x^4$. Thus, $f(x) = f(-x)$ and $f(x)$ is even.

(b) $g(-x) = (-x)^{-1} = -x^{-1}$. Thus, $g(-x) = -g(x)$, and $g(x)$ is odd.

(c) $h(-x) = (-x)^2 + (-x) = x^2 - x$. We see that $h(-x)$ is not equal to $h(x)$ or to $-h(x) = -x^2 - x$. Therefore, $h(x)$ is neither even nor odd.

EXAMPLE **Using Symmetry** Sketch the graph of $f(x) = \dfrac{1}{x^2+1}$.

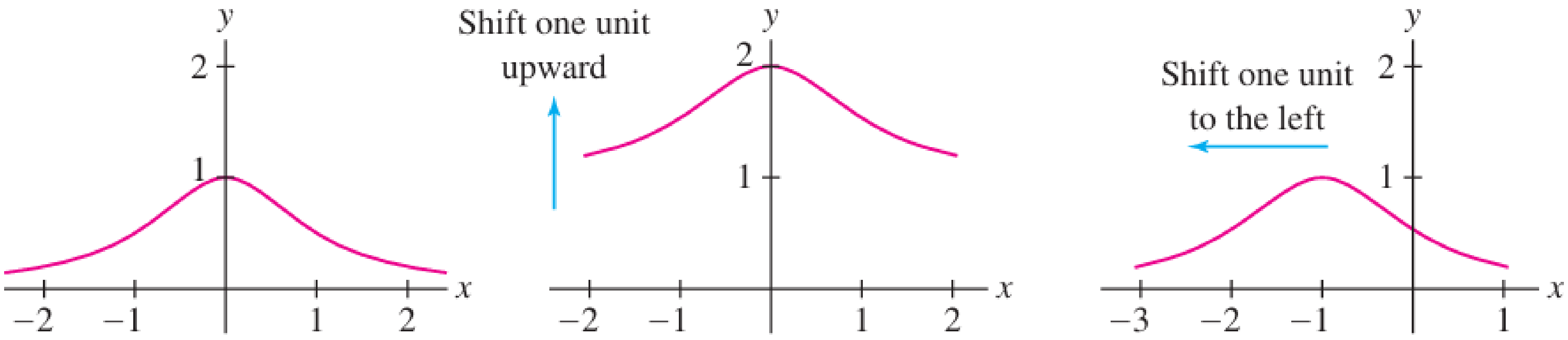
x	$\dfrac{1}{x^2+1}$
0	1
± 1	$\frac{1}{2}$
± 2	$\frac{1}{5}$



DEFINITION Translation (Shifting)

- **Vertical translation** $y = f(x) + c$: shifts the graph by $|c|$ units *vertically*, upward if $c > 0$ and c units downward if $c < 0$.
- **Horizontal translation** $y = f(x + c)$: shifts the graph by $|c|$ units *horizontally*, to the right if $c < 0$ and c units to the left if $c > 0$.

Figure shows the effect of translating the graph of $f(x) = 1/(x^2 + 1)$ vertically and horizontally.



(A) $y = f(x) = \dfrac{1}{x^2 + 1}$ (B) $y = f(x) + 1 = \dfrac{1}{x^2 + 1} + 1$ (C) $y = f(x + 1) = \dfrac{1}{(x + 1)^2 + 1}$

DEFINITION Scaling

- **Vertical scaling** $y = kf(x)$: If $k > 1$, the graph is expanded vertically by the factor k. If $0 < k < 1$, the graph is compressed vertically. When the scale factor k is negative ($k < 0$), the graph is also reflected across the x-axis (Figure 24).
- **Horizontal scaling** $y = f(kx)$: If $k > 1$, the graph is compressed in the horizontal direction. If $0 < k < 1$, the graph is expanded. If $k < 0$, then the graph is also reflected across the y-axis.

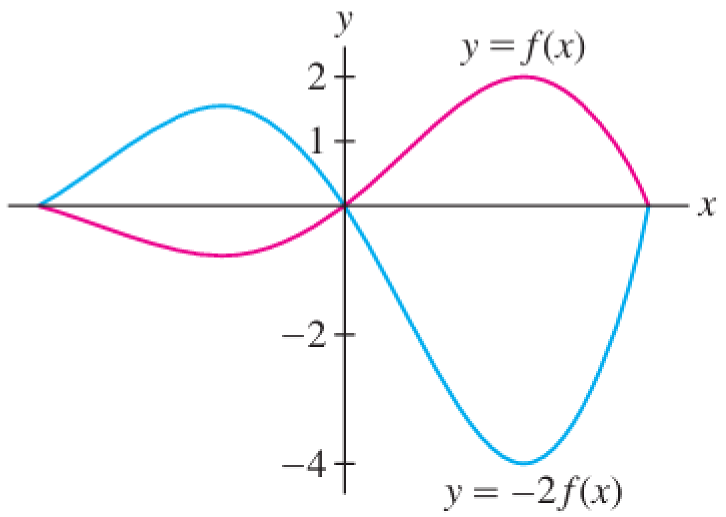
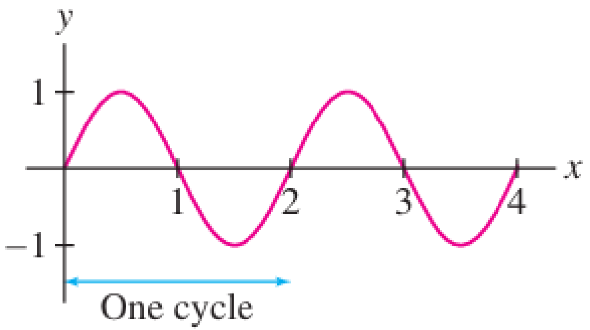


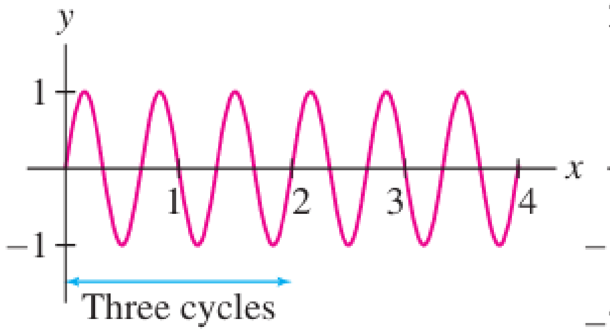
FIGURE Negative vertical scale factor $k = -2$.

EXAMPLE Sketch the graphs of $f(x) = \sin(\pi x)$ and its dilates $f(3x)$ and $3f(x)$.

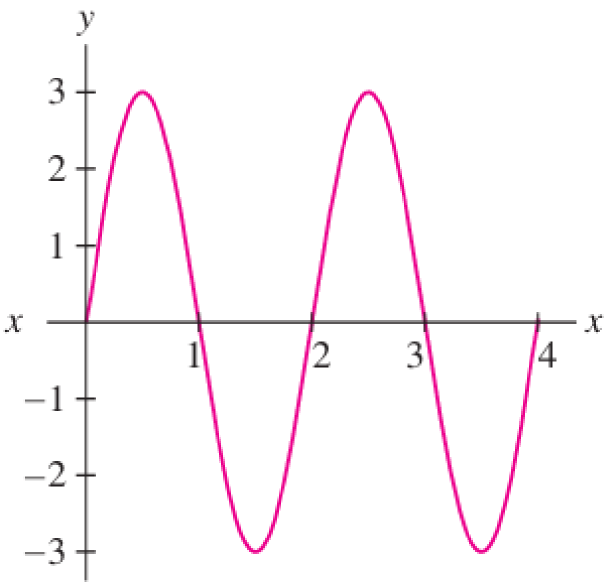
Solution



(A) $y = f(x) = \sin(\pi x)$



(B) Horizontal compression:
$y = f(3x) = \sin(3\pi x)$



(C) Vertical expansion:
$y = 3f(x) = 3\sin(\pi x)$